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Strongly Chebyshev Subspaces of Matrices

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1. INTRODUCTION

Hilbert spaces have a unique position in abstract approximation theory. For example, every closed linear subspace of a Hilbert space is a Chebyshev subspace and the proximity map is even linear, being given by the orthogonal projection map. A Hilbert space can, however, be isometrically embedded in a normed linear space which is less well behaved from this point of view and the question arises of how elements in the larger space can be approximated from the Hilbert space. We shall be concerned with finite dimensional real Hilbert spaces (i.e., Euclidean spaces) which are naturally embedded in the space $M_n(\mathbb{R})$ of real $n \times n$ matrices, endowed with the spectral norm. Such embeddings arise naturally in geometry when $M_n(\mathbb{R})$ is regarded as the Clifford algebra of a Euclidean space E [4, Chap. 13]. We shall show that E is then a Chebyshev subspace of $M_n(\mathbb{R})$.

Our result holds for a slightly more general class of subspaces of $M_n(\mathbb{R})$, but the phenomenon is essentially restricted to *real* matrices, thus demonstrating a purely geometric difference between $M_n(\mathbb{R})$ and the normed linear space $M_n(\mathbb{C})$ of complex $n \times n$ matrices. The main result allows us to characterize the strongly Chebyshev subspaces of $M_n(\mathbb{R})$, these being, by definition, those subspaces $\mathscr{V}$ having the property that every subspace of $\mathscr{V}$ is Chebyshev in $M_n(\mathbb{R})$. A classical result in topology [1] can then be interpreted as specifying the maximum possible dimension of a strongly Chebyshev subspace of $M_n(\mathbb{R})$.

2. LINEAR SUBSPACES OF $GL(n, \mathbb{R})$

In what follows $M_n(\mathbb{R})$ is regarded as a normed linear space with the spectral norm induced by the usual Euclidean norm on $\mathbb{R}^n$ [3]. Thus $\|A\|$

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will denote the spectral norm of a matrix and $\|\mathbf{v}\| = (\mathbf{v}^T\mathbf{v})^{1/2}$ the Euclidean norm of a vector $\mathbf{v} \in \mathbb{R}^n$. $GL(n; \mathbb{R})$ denotes the set of nonsingular matrices in $M_n(\mathbb{R})$. A *linear subspace of* $GL(n; \mathbb{R})$ is a linear subspace $\mathscr{V}$ of $M_n(\mathbb{R})$ having the property that every nonzero matrix in $\mathscr{V}$ is nonsingular [4, p. 272]. Such subspaces arise naturally in algebraic topology in connection with linearly independent vector fields on spheres [4, Theorem 20.68]. Particularly important examples of linear subspaces of $GL(n; \mathbb{R})$ are what we shall call *Clifford subspaces*. These have the property that $X^TX = \|X\|^2 I$ for all $X \in \mathscr{V}$, and are constructed in [4, Prop. 13.67]. Simple examples of Clifford subspaces are provided by the usual embedding of the complex numbers as a linear subspace of $M_2(\mathbb{R})$ or of the quaternions in $M_4(\mathbb{R})$. A Clifford subspace $\mathscr{V}$ of $M_n(\mathbb{R})$ is a Hilbert space in the spectral norm, the inner product being defined on $\mathscr{V}$ by the identity

$$X^TY + Y^TX = 2(X, Y)\, I.$$

Our first result demonstrates that such subspaces are of interest in approximation theory. Recall that a linear subspace $\mathscr{V}$ of a normed linear space is a Chebyshev subspace if every vector has a unique best approximant from $\mathscr{V}$.

THEOREM 1. *Let* $\mathscr{V}$ *be a linear subspace of* $GL(n; \mathbb{R})$. *Then* $\mathscr{V}$ *is a Chebyshev subspace of* $M_n(\mathbb{R})$. *If* $\mathscr{V}$ *is a Clifford subspace and* 0 *is the best approximation to* $A \in M_n(\mathbb{R})$ *from* $\mathscr{V}$ *then the following Pythagorean relation holds for all* $X \in \mathscr{V}$:

$$\|A - X\|^2 \geqslant \|A\|^2 + \|X\|^2.$$

Proof. By finite dimensionality best approximations always exist, so we need only consider the question of uniqueness. It is enough to show that if $A \in M_n(\mathbb{R})$ has 0 as a best approximant from $\mathscr{V}$ then 0 is the unique best approximant.

According to [3, Theorem 3], there exist positive scalars $\lambda_1, ..., \lambda_k$ with $\sum \lambda_j = 1$ and unit vectors $\mathbf{u}_1, ..., \mathbf{u}_k, \mathbf{v}_1, ..., \mathbf{v}_k$, such that

$$\sum \lambda_j \mathbf{u}_j^T X \mathbf{v}_j = 0, \qquad \text{for all } X \in \mathscr{V}$$

and

$$\mathbf{u}_j^T A \mathbf{v}_j = \|A\|, \qquad \text{for} \quad 1 \leqslant j \leqslant k.$$

The latter condition implies that we have equality in the Cauchy–Schwarz inequality, from which it follows that $A\mathbf{v}_j = \|A\|\, \mathbf{u}_j$, for $1 \leqslant j \leqslant k$. Also if $X \in \mathscr{V}$ and $X \neq 0$ then

$$\sum \lambda_j \mathbf{v}_j^T A^T X \mathbf{v}_j = \sum \lambda_j \|A\|\, \mathbf{u}_j^T X \mathbf{v}_j = 0.$$

Hence

$$\begin{aligned}\|A-X\|^2 &\geqslant \sum \lambda_j \,\|(A-X)\,\mathbf{v}_j\|^2\\ &=\sum \lambda_j \mathbf{v}_j^T(A-X)^T(A-X)\,\mathbf{v}_j\\ &=\sum \lambda_j \mathbf{v}_j^T A^T A \mathbf{v}_j+\sum \lambda_j \mathbf{v}_j^T X^T X \mathbf{v}_j\\ &=\sum \lambda_j \,\|A\|^2\, \mathbf{u}_j^T \mathbf{u}_j+\sum \lambda_j \|X\mathbf{v}_j\|^2\\ &=\|A\|^2+\sum \lambda_j \,\|X\mathbf{v}_j\|^2\\ &>\|A\|^2,\end{aligned}$$

since $X \in \mathscr{V}\backslash\{0\}$, so that X is nonsingular. This shows that 0 is the unique best approximant to A.

Finally, if $\mathscr{V}$ is a Clifford subspace then $\|X\mathbf{v}_j\|^2=\|X\|^2$ for $1 \leqslant j \leqslant k$, so that $\|A-X\|^2 \geqslant \|A\|^2+\|X\|^2$. ▮

If $\mathscr{V}$ is a Clifford subspace of $M_n(\mathbb{R})$, we therefore have a well-defined metric projection π: $M_n(\mathbb{R}) \rightarrow \mathscr{V}$ onto $\mathscr{V}$ for each $A \in M_n(\mathbb{R})$. The following continuity property of π follows from the Pythagorean relation in Theorem 1 by exactly the same calculations as in [7, Theorem 3]. We omit the details.

COROLLARY 2. *If $\mathscr{V}$ is a Clifford subspace of $M_n(\mathbb{R})$ then the corresponding metric projection π satisfies*

$$2\|\pi(A)-\pi(B)\| \leqslant \delta+(\delta^2+8\delta\|A\|)^{1/2},$$

where A, $B \in M_n(\mathbb{R})$ and $\delta=\|A-B\|$.

Direct calculation shows that the diagonal matrices form a Chebyshev subspace of $M_2(\mathbb{R})$, so that the converse of Theorem 1 is false. However, in the one-dimensional case, the converse of Theorem 1 is true.

COROLLARY 3. *Let X be a nonzero matrix in $M_n(\mathbb{R})$. $\mathbb{R}X$ is a Chebyshev subspace of $M_n(\mathbb{R})$ if and only if X is nonsingular.*

Proof. $\mathbb{R}X$ is a Chebyshev subspace if X is nonsingular, by Theorem 1. On the other hand suppose that X is singular. Then there exist unit vectors

$\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ such that $X\mathbf{u} = 0$ and $\mathbf{v}^T X = 0$. Let $Y = \mathbf{v}\mathbf{u}^T$. Then $XY^T = Y^T X = 0$. Therefore, for each $\lambda \in \mathbb{R}$,

$$\begin{aligned} \|Y - \lambda X\|^2 &= \|(Y^T - \lambda X^T)(Y - \lambda X)\| \\ &= \|Y^T Y + \lambda^2 X^T X\| \\ &= \max(\|Y^T Y\|, \lambda^2 \|X^T X\|), \end{aligned}$$

where the last equality follows from the fact that $(X^T X)(Y^T Y) = 0$. It is clear from this that $\|Y - X\|$ is constant for small values of λ, so that $\mathbb{R}X$ is not a Chebyshev subspace ∎

Remark. In [5, Theorem 2.8] a complex infinite dimensional version of Theorem 1 was proved by different methods. However that complex result becomes trivial in the finite dimensional case in view of the fact that there do not exist linear subspaces of $GL(n; \mathbb{C})$ of dimension greater than one. For if $\mathscr{V}$ were such a subspace of $GL(n; \mathbb{C})$, choose linearly independent matrices X, $Y \in \mathscr{V}$. If $\mu \in \mathbb{C}$ is an eigenvalue of $Y^{-1}X$ then $X - \mu Y = Y(Y^{-1}X - \mu I)$ is singular, which is a contradiction since $X - \mu Y$ is a non-zero element of $\mathscr{V}$.

3. Strongly Chebyshev Subspaces

Let us call a linear subspace $\mathscr{V}$ of a normed linear space $\mathscr{W}$ a *strongly Chebyshev subspace* if every closed linear subspace of $\mathscr{V}$ (including $\mathscr{V}$ itself) is a Chebyshev subspace of $\mathscr{W}$. In the finite dimensional case this is equivalent to requiring that every one-dimensional subspace of $\mathscr{V}$ is Chebyshev in $\mathscr{W}$, since in that case best approximants always exist and if X, Y are distinct best approximants to an element of $\mathscr{W}$ then $\mathbb{R}(X - Y)$ is not Chebyshev in $\mathscr{W}$. Clearly every closed linear subspace of a strongly Chebyshev subspace is again strongly Chebyshev. Also every strongly Chebyshev subspace of a normed linear space is strictly convex [6, Chap. I, Corollary 3.3].

THEOREM 4. *A linear subspace $\mathscr{V}$ of $M_n(\mathbb{R})$ is a strongly Chebyshev subspace if and only if $\mathscr{V}$ is a linear subspace of $GL(b; \mathbb{R})$.*

Proof. This is an immediate consequence of Theorem 1 and Corollary 3. ∎

For each positive integer n the maximum possible dimension of a linear subspace of $GL(n; \mathbb{R})$ is equal to the Hurwitz–Radon number $\rho(n)$ [1, 2]. Moreover this maximum dimension is attained for some Clifford subspace of $\mathscr{V}$ of $M_n(\mathbb{R})$ [4, Theorem 13.68]. Recall that a Clifford subspace is, in particular, a real Hilbert space embedded in $M_n(\mathbb{R})$.

We can now write down a direct interpretation of these statements purely in terms of the geometry of $M_n(\mathbb{R})$.

THEOREM 5. *The maximum dimension of a strongly Chebyshev subspace $\mathscr{V}$ of $M_n(\mathbb{R})$ is $\rho(n)$. This maximum is attained for a Hilbert space $\mathscr{V}$.*

Remarks. (1) A strongly Chebyshev subspace of $M_n(\mathbb{R})$ need not be a Hilbert space. For example, consider all matrices of the form $\left[\begin{smallmatrix} a & b \\ -2b & a \end{smallmatrix}\right]$ in $M_2(\mathbb{R})$.

(2) If the positive integer n is written as an odd multiple of 2^{a+4b}, where a, b are integers and $0 \leqslant a \leqslant 3$, then by definition $\rho(n) = 2^a + 8b$. Thus $\rho(n) = 1$ if n is odd, so in that case there is no strongly Chebyshev subspace of $M_n(\mathbb{R})$ of dimension greater than one. Other values of $\rho(n)$ for small n are $\rho(2) = 2$, $\rho(4) = 4$, $\rho(6) = 2$, $\rho(8) = 8$; so, for example, the natural embedding of the quaternions in $M_4(\mathbb{R})$ provides us with an example of a strongly Chebyshev subspace of maximal dimension.

(3) As previously noted, there is no linear subspace of $GL(n; \mathbb{C})$ of dimension greater than one. However the complex versions of Theorem 1 and Corollary 3 are valid. Therefore $M_n(\mathbb{C})$ can contain no strongly Chebyshev subspace of dimension greater than one, thus providing a curious geometrical contrast with the case of $M_n(\mathbb{R})$. On the other hand it was shown in [5, Theorem 2.8] that the algebra of bounded linear operators on an infinite dimensional complex Hilbert space contains an infinite dimensional strongly Chebyshev subspace.

Finally, it is natural to ask whether $\rho(n)$ provides an upper bound on the dimension of a Hilbert space embedded in $M_n(\mathbb{R})$. The answer is no.

EXAMPLE. Given $a, b \in \mathbb{R}$, define $X \in M_3(\mathbb{R})$ by

$$X = \begin{bmatrix} 0 & a & 0 \\ a & 0 & b \\ 0 & b & 0 \end{bmatrix}.$$

Then $\|X\|^2 = a^2 + b^2$, so that the set of such matrices forms a two-dimensional Hilbert space $\mathscr{E}$ embedded in $M_3(\mathbb{R})$. (Recall that $\rho(3) = 1$.)

We can modify our original question and ask if $\rho(n)$ is an upper bound on the dimension of a Chebyshev subspace of $M_n(\mathbb{R})$ which is also a Hilbert space. We do not know the answer. Of course there do exist Chebyshev subspaces of $M_n(\mathbb{R})$ of dimension greater than $\rho(n)$. An explicit example is the set of all matrices with zero trace.

REFERENCES

1. J. F. ADAMS, Vector fields on spheres, *Ann. of Math.* (*2*) **75** (1962), 603–632.
2. J. F. ADAMS, P. D. LAX, AND R. S. PHILLIPS, On matrices whose real linear combinations are nonsingular, *Proc. Amer. Math. Soc.* **16** (1965), 318–322.
3. K. K. LAU AND W. O. RIHA, Characterization of best approximations in normed linear spaces of matrices by elements of finite-dimensional linear subspaces, *Linear Algebra Appl.* **35** (1981), 109–120.
4. I. R. PORTEOUS, "Topological Geometry," Van Nostrand–Reinhold, London, 1969.
5. A. G. ROBERTSON AND D. T. YOST, Chebyshev subspaces of operator algebras, *J. London Math. Soc* (*2*) **19** (1979), 523–531.
6. I. SINGER, "Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces," Springer-Verlag New York/Berlin, 1970.
7. J. G. STAMPFLI, The norm of a derivation, *Pacific J. Math.* **33** (1970), 737–747.